An integration problem from Malaysia student

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The following is an integration problem send to me by a Malaysia Pre-U student:

Evaluate : $\int \frac{5\cos x+3\sin x}{3\cos x-\sin x}dx$

I begin with the standard method of *t-substitution.*

Method 1

$$I=\int \frac{5\cos x+3\sin x}{3\cos x-\sin x}dx$$

Let $t=\tan\frac{x}{2}$, $\sin x=\frac{2t}{1+t^2}$, $\cos x=\frac{1-t^2}{1+t^2}$, $dx=\frac{2dt}{1+t^2}$

$$I=\int \frac{5\left(\frac{1-t^2}{1+t^2}\right)+3\left(\frac{2t}{1+t^2}\right)}{3\left(\frac{1-t^2}{1+t^2}\right)-\left(\frac{2t}{1+t^2}\right)}\frac{2dt}{1+t^2}=\int \frac{10t^2-12t-10}{(t^2+1)(3t^2+2t-3)}dt$$

$$=\int\left[\frac{2(7t+6)}{5(t^2+1)}-\frac{14(3t+1)}{5(3t^2+2t-3)}\right]dt$$ *(Partial fraction: See appendix 1 below.)*

$$=\frac{7}{5}\int\frac{d(t^2+1)}{t^2+1}+\frac{12}{5}\int\frac{dt}{t^2+1}-\frac{7}{5}\int\frac{d(3t^2+2t-3)}{3t^2+2t-3}$$

$$=\frac{7}{5}\ln(t^2+1)+\frac{12}{5}\tan^{-1}t-\frac{7}{5}\ln(3t^2+2t-3)+C$$

$$=\frac{7}{5}\ln\left(\tan^2\frac{x}{2}+1\right)+\frac{12}{5}\left(\frac{x}{2}\right)-\frac{7}{5}\ln\left(3\tan^2\frac{x}{2}+2\tan\frac{x}{2}-3\right)+C$$

$$=\frac{6}{5}x-\frac{7}{5}\ln\left(\frac{3\tan^2\frac{x}{2}+2\tan\frac{x}{2}-3}{\tan^2\frac{x}{2}+1}\right)+C=\frac{6}{5}x-\frac{7}{5}\ln\left|\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}-3\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right|+C$$

$$=\mathbf{\frac{6}{5}x-\frac{7}{5}\ln|\sin x-3\cos x|+c}$$

As the reader can see, it works, but the calculation is tedious. So I begin another way.

Method 2

$$I=\int \frac{5\cos x+3\sin x}{3\cos x-\sin x}dx=\int\frac{5+3\frac{\sin x}{\cos x}}{3-\frac{\sin x}{\cos x}}dx=\int\frac{5+3\tan x}{3-\tan x}dx$$

Put $t=\tan x$, $dt=\sec^2 x\,dx=(1+\tan^2 x)dx$, $\frac{dt}{1+t^2}=dx$

$\therefore \mathrm{I} = \int \frac{5+3t}{3-t}\frac{dt}{1+t^2} = -\int \frac{3t+5}{(t^2+1)(t-3)}dt$

$= \left[\int \frac{7t+6}{5(t^2+1)} - \frac{7}{5(t-3)}\right]dt$ *(Partial fraction: See appendix 2 below.)*

$= \frac{7}{10}\int \frac{d(t^2+1)}{t^2+1} + 6\int \frac{dt}{t^2+1} - \frac{7}{5}\int \frac{dt}{t-3} = \frac{7}{10}\ln(t^2+1) + \frac{6}{5}\tan^{-1}t - \frac{7}{5}\ln(t-3) + c$

$= \frac{7}{10}\ln|\tan^2 x + 1| + \frac{6}{5}x - \frac{7}{5}\ln|\tan x - 3| + c$

$= \frac{7}{10}\ln|\sec^2 x| + \frac{6}{5}x - \frac{7}{5}\ln|\tan x - 3| + c$

$= \frac{7}{5}\ln|\sec x| + \frac{6}{5}x - \frac{7}{5}\ln|\tan x - 3| + c$

$= \frac{6}{5}x - \frac{7}{5}\ln\left|\frac{\tan x - 3}{\sec x}\right| + C = \mathbf{\frac{6}{5}x - \frac{7}{5}\ln|\sin x - 3\cos x| + c}$

Although the partial fraction is easier than Method 1, there is quite a lot of work to do.
Method 3 below needs keen observation, quite nice.

Method 3

Since $d(3\cos x - \sin x) = (-3\sin x - \cos x)dx$

Let $5\cos x + 3\sin x = A(3\cos x - \sin x) + B(-3\sin x - \cos x) = (3A - B)\cos x + (-A - 3B)\sin x$

Therefore $3A - B = 5$

$-A - 3B = 3$

Solving, we have $A = \frac{6}{5}$, $B = -\frac{7}{5}$

$5\cos x + 3\sin x = \frac{6}{5}(3\cos x - \sin x) - \frac{7}{5}(-3\sin x - \cos x)$

$\int \frac{5\cos x + 3\sin x}{3\cos x - \sin x}dx = \int \frac{\frac{6}{5}(3\cos x - \sin x) - \frac{7}{5}(-3\sin x - \cos x)}{3\cos x - \sin x}dx = \frac{6}{5}\int dx - \frac{7}{5}\int \frac{d(3\cos x - \sin x)}{3\cos x - \sin x}$

$= \mathbf{\frac{6}{5}x - \frac{7}{5}\ln|3\cos x - \sin x| + c}$

Method 4 begins by inventing two integrals, very enjoyable.

Method 4

$I_1 = \int \frac{\cos x}{3\cos x - \sin x}dx, \qquad I_2 = \int \frac{\sin x}{3\cos x - \sin x}dx$

For simplicity, we don't write integrating constants until the last step.

Then $3I_1 - I_2 = \int \frac{3\cos x - \sin x}{3\cos x - \sin x}\,dx = \int dx = x \quad \dots(1)$

Also, $\int \frac{d(3\cos x - \sin x)}{3\cos x - \sin x} = \int \frac{-3\sin x - \cos x}{3\cos x - \sin x}\,dx = -I_1 - 3I_2$

Hence, $-I_1 - 3I_2 = \ln|3\cos x - \sin x| \quad \dots(2)$

Solving (1) and (2), $I_1 = \frac{3}{10}x - \frac{1}{10}\ln|3\cos x - \sin x|$

$$I_2 = -\frac{1}{10}x - \frac{3}{10}\ln|3\cos x - \sin x|$$

$\int \frac{5\cos x + 3\sin x}{3\cos x - \sin x}\,dx = 5I_1 + 3I_2 = \mathbf{\frac{6}{5}x - \frac{7}{5}\ln|\sin x - 3\cos x| + c}$

Generalization of the integral

$$\int \frac{\mathbf{a\cos x + b\sin x}}{\mathbf{c\cos x + d\sin x}}\,\mathbf{dx} = \mathbf{\frac{ac+bd}{c^2+d^2}x + \frac{ad-bc}{c^2+d^2}\ln|c\cos x + d\sin x| + constant}$$

(You may try using method 3 or 4, not too difficult!)

Appendix

1. $\frac{10t^2 - 12t - 10}{(t^2+1)(3t^2+2t-3)} \equiv \frac{At+B}{t^2+1} + \frac{Ct+D}{3t^2+2t-3}$

$10t^2 - 12t - 10 \equiv (At+B)(3t^2+2t-3) + (Ct+D)(t^2+1)$
$\equiv (3A+C)t^3 + (2A+3B+D)t^2 + (-3A+2B+C)t + (-3B+D)$

Hence,
$3A + C = 0$
$2A + 3B + D = 10$
$-3A + 2B + C = -12$
$-3B + D = -10$

Solving, $A = \frac{14}{5},\ B = \frac{12}{5},\ C = -\frac{42}{5},\ D = -\frac{14}{5}$

$$\therefore \frac{10t^2 - 12t - 10}{(t^2+1)(3t^2+2t-3)} \equiv \frac{2(7t+6)}{5(t^2+1)} - \frac{14(3t+1)}{5(3t^2+2t-3)}$$

2. $-\frac{3t+5}{(t^2+1)(t-3)} \equiv \frac{At+B}{t^2+1} + \frac{C}{t-3}$

$-(3t+5) \equiv (At+B)(t-3) + C(t^2+1)$

Put $t = 3$, $-14 \equiv 10C$, $C = -\frac{7}{5}$

Hence, $-(3t+5) \equiv (At+B)(t-3) - \frac{7}{5}(t^2+1) \equiv \left(A - \frac{7}{5}\right)t^2 + (B-3A)t - 3B - \frac{7}{5}$

Compare coefficient on t^2-term, $0 = A - \frac{7}{5}$, $A = \frac{7}{5}$

Compare coefficient on constant term, $-5 = -3B - \frac{7}{5}$, $B = \frac{6}{5}$

$$\therefore -\frac{3t+5}{(t^2+1)(t-3)} \equiv \frac{7t+6}{5(t^2+1)} - \frac{7}{5(t-3)}$$